Indian Statistical Institute

Midterm Exam. 2023-2024

Measure Theory, M.Math First Year

Time : 3 Hours   Date : 15.09.2023   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) You may freely apply any of the theorems we discussed in class. (iii) $m =$ the Lebesgue measure on $\mathbb{R}$.

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(1) (15 marks) Give an example of a finite measure space that is not complete. Determine the completion of the measure space.

(2) (15 marks) Let E be a Lebesgue measurable subset of $\mathbb{R}$. If $m(E) < \infty$, then prove that

$$m\Big(E \cap [r, \infty)\Big) \longrightarrow 0,$$

as $r \to \infty$.

(3) (15 marks) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Prove that the derivative $f' : \mathbb{R} \to \mathbb{R}$ is measurable.

(4) (15 marks) Let $(X, \mathcal{A})$ be a measurable space, and let $\{f_n\}_{n \geq 1}$ be a sequence of real-valued measurable functions. Prove that

$$\{x \in X : \{f_n(x)\}_{n\geq 1} \text{ is eventually constant}\},$$

is a measurable set.

(5) (15 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, that is

$$|f(x) - f(y)| \leq M|x - y|,$$

for all x and y, where M is a constant independent of x and y. Suppose E is a set of Lebesgue measure zero. Prove that $f(E)$ also has Lebesgue measure zero.

(6) (15 marks) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Suppose $\epsilon > 0$. Prove that there exists $\alpha > 0$ such that

$$\mu\Big(\{x \in X : |f(x)| > \alpha\}\Big) < \epsilon.$$

(7) (20 marks) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $\mathcal{D}$ be a set of disjoint measurable subsets of X. Suppose

$$\mu(D) > 0,$$

for all $D \in \mathcal{D}$. Prove that $\mathcal{D}$ is at most a countable set.